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LETTER TO THE EDITOR

Quantum group symmetry in multiple Chern–Simons theory on a torus

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Abstract. We discuss w_∞ and $sl_q(2)$ symmetries in multiple Chern–Simons theory on a torus. It is shown that these algebraic structures arise from the dynamics of the non-integrable phases of the Chern–Simons fields. The generators of these algebras are constructed from the Wilson line operators corresponding to these phases. The vacuum states form the basis of cyclic representation of $sl_q(2)$.

In the last decade the studies of quantum groups and algebras have attracted a lot of interest. One can already see the great impact these studies have had in physics and mathematics. The concept of quantum groups and algebras has its origin in the development of the quantum inverse method and the study of solutions to the Yang–Baxter equation [1]. These new mathematical structures have already found applications in exactly solvable statistical models, in two-dimensional conformal field theory and in non-Abelian Chern–Simons (CS) theory [2]. Undoubtedly, it is of interest to find applications of these concepts in other important systems, especially realistic ones.

Recently a $sl_q(2)$ quantum group symmetry has been uncovered in the Landau problem (i.e. a charged particle moving in a constant magnetic field) and in the related problem of fractional quantum Hall effects (FQHE) [3–8]. A quantum w_∞ algebra, also known as the FFZ [9] algebra, has also been realized in these systems [4, 5, 10, 11]. A representation of the quantum algebra $sl_q(2)$ was applied to formulate the Bethe ansatz for the problem of a Bloch electron in a magnetic field, i.e. the Azbel–Hofstadter problem [3, 12, 13].

On the other hand, Abelian Chern–Simons field theory with matter coupling has attracted intense interest in recent years, owing to its relevance to condensed matter systems such as quantum Hall systems, and possibly high- T_c superconductors. Many studies have also been carried out for the Maxwell–Chern–Simons theory in which a Maxwell kinetic term is included. An interesting observation is that the dynamics in the topological sector of Maxwell–Chern–Simons theory on a torus is equivalent to the Landau problem on a torus. Thus many interesting features are shared by both systems. In fact, the quantum algebras mentioned previously were also found in the Maxwell and the pure CS theory on a torus [5, 7].

In this letter we would like to extend these results to a theory on a torus in which multiple kinds of Chern–Simons gauge interactions are introduced among particles [14]. It is known that multiple Chern–Simons interactions induce matrix statistics which generalize ordinary fractional statistics in the space of particle species. A possible application of the theory is in double-layered Hall systems [14, 15].

Let us consider a theory on a torus (of lengths L_1 and L_2) with M distinct CS gauge fields, a_μ^I ($I = 1, \dots, M$) and nonrelativistic matter fields. The theory is described by the Lagrangian $\mathcal{L} = \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{matter}}$, where the Chern–Simons term \mathcal{L}_{CS} is given by

$$\mathcal{L}_{\text{CS}} = \frac{1}{4\pi} \sum_{IJ} K_{IJ} a^I \varepsilon \partial a^J \quad (I, J = 1, \dots, M) \quad (1)$$

$$(\varepsilon \partial a)^\mu \equiv \varepsilon^{\mu\nu\rho} \partial_\nu a_\rho.$$

Here K_{IJ} is a $M \times M$ real symmetric matrix. Properties of this theory are studied in [16], and we refer the reader to this reference for details. Only those features relevant to the present discussions are summarized below.

On the torus, for each CS field there are two non-integrable phases, θ_j^I ($j = 1, 2$), of the Wilson line integrals along the two non-contractible loops of the torus. These phases are new degrees of freedom undetermined by the matter content, and their contributions are found to be decoupled in the action of the theory. They are responsible for the topological structures of the theory. We will be interested only in the structures of the Hilbert space of these Wilson line phases in the rest of the letter.

In [16] it has been found that the Lagrangian corresponding to the Wilson line phases is of the form $K_{IJ}(\theta_2^I \dot{\theta}_1^J - \theta_1^I \dot{\theta}_2^J)/4\pi$. This implies that θ_1^I 's and θ_2^I 's are conjugate pairs:

$$[\theta_1^I, \theta_2^J] = 2\pi i K_{IJ}^{-1} \quad (2)$$

where K_{IJ}^{-1} is the IJ -component of the matrix K^{-1} . The system is invariant under large gauge transformations which shift the Wilson line phases by multiples of 2π :

$$U_j^I : \theta_j^I \rightarrow \theta_j^I + 2\pi. \quad (3)$$

Unitary operators inducing the transformation (3) are given by

$$U_j^I = e^{+i\varepsilon^{jk} K_{IJ} \theta_k^I}. \quad (4)$$

The two sets of operators, $\{U_j^I\}$ and $\{W_j^I\}$, are complimentary. They satisfy the Weyl–Heisenberg (WH) relations:

$$\begin{aligned} U_1^I U_2^J &= e^{-2\pi i K_{IJ}} U_2^J U_1^I \\ W_1^I W_2^J &= e^{-2\pi i K_{IJ}^{-1}} W_2^J W_1^I \\ U_j^I W_k^J &= W_k^J U_j^I. \end{aligned} \quad (5)$$

Note that these operators do not commute with each other in general. The algebra is invariant under the interchange of U_j^I and W_j^I supplemented by the replacement of K_{IJ} by K_{IJ}^{-1} . This suggests that there is a duality between the theories with the Chern–Simons coefficient matrix K and with K^{-1} .

We would like to determine vacuum wavefunctions that form a representation of the WH group (5). From now on we shall suppose that all K_{IJ} 's are integers so that all the U_j^I 's commute among themselves. We may thus simultaneously diagonalize these operators and take

$$U_j^I |\Psi\rangle = e^{i\gamma_j^I} |\Psi\rangle \quad (6)$$

where γ_j^I are the vacuum angles. For convenience we introduce vector notation: $\boldsymbol{\theta}_j = (\theta_j^1, \dots, \theta_j^M)$, $\boldsymbol{\gamma}_j = (\gamma_j^1, \dots, \gamma_j^M)$, etc. It has been shown that the degeneracy of vacua is $r = \det K$ [16, 17]. An independent basis of vacua $|\mathbf{h}_a\rangle$ can be chosen as (in the $\boldsymbol{\theta}_1$ representation) [16]:

$$(\boldsymbol{\theta}_1 | \mathbf{h}_a) \equiv u_a(\boldsymbol{\theta}_1) = e^{i\boldsymbol{\gamma}_1 \cdot \boldsymbol{\theta}_1 / 2\pi} \delta_{2\pi}[\boldsymbol{\theta}_1 + K^{-1} \boldsymbol{\gamma}_2 - \mathbf{h}_a] \quad (a = 1, \dots, r = \det K). \quad (7)$$

The set of vectors $\mathcal{H}(K) = \{\mathbf{h}_a\}$ is defined by

$$\mathcal{H}(K) = \{\mathbf{h}_a \in R^M, (a = 1, \dots, r); K\mathbf{h}_a \sim 0\} \quad (8)$$

where the equivalence relation \sim among vectors $\in R^M$ is defined by:

$$\mathbf{h} \sim \mathbf{g} \iff h^I = g^I \pmod{2\pi} \quad I = 1, \dots, M. \quad (9)$$

Vectors in $\mathcal{H}(K)$ are independent in the sense that $\mathbf{h}_a \not\sim \mathbf{h}_b$ iff $a \neq b$.

The actions of the Wilson lines on the vacuum are:

$$\begin{aligned} W_1^I |\mathbf{h}_a\rangle &= e^{-il_I \cdot \gamma_2 - ih_a^I} |\mathbf{h}_a\rangle \\ W_2^I |\mathbf{h}_a\rangle &= e^{+il_I \cdot \gamma_1} |\mathbf{h}_a - 2\pi l_I\rangle. \end{aligned} \quad (10)$$

Here l_I are the column vectors of the matrix K^{-1} : $K^{-1} = (l_1, \dots, l_M)$. Note that, if \mathbf{k}_I are the column vectors of K : $K = (\mathbf{k}_1, \dots, \mathbf{k}_M)$, then we have the orthogonality relation: $\mathbf{k}_I \cdot \mathbf{l}_J = \delta_{IJ}$. It is easy to see that W_2^I induces a mapping among the vacua. In fact, we have $K(\mathbf{h}_a - 2\pi l_I) \sim 0$, so that $\mathbf{h}_a - 2\pi l_I \in \mathcal{H}$. If there exists a K_{IJ}^{-1} such that $\exp(2\pi i K_{IJ}^{-1})$ is a primitive r th root of unity, i.e. r is the least integer such that $q^r = 1$, then $(W_2^J)^r \sim I$, and W_2^J maps all the r distinct vacua among themselves.

We can now reveal the quantum algebraic structures inherent in the theory. First let us form, from the W_j^I , the following operators:

$$T_n = T_{(n_1, n_2)} \equiv q^{n_1 n_2 / 2} (W_1^I)^{n_1} (W_2^J)^{n_2} \quad (11)$$

where $q \equiv e^{2\pi i K_{IJ}^{-1}}$, and n_1, n_2 are integers. Note here that there are $M \times M$ possible sets of operators T , depending on the choice of I and J . From the WH algebras (5) one gets:

$$T_m T_n = q^{-m \times n / 2} T_{m+n} \quad (12)$$

where $m \times n = m_1 n_2 - m_2 n_1$. Equation (12) implies

$$[T_m, T_n] = -2i \sin(i\pi K_{IJ}^{-1} (m \times n)) T_{m+n}. \quad (13)$$

This is nothing but the quantum w_∞ (FFZ) algebra.

Next we construct the operators J_\pm and J_3 from the T 's as follows [4-7]:

$$\begin{aligned} J_+ &\equiv \frac{1}{q - q^{-1}} (T_{(1,1)} - T_{(-1,1)}) \\ J_- &\equiv \frac{1}{q - q^{-1}} (T_{(-1,-1)} - T_{(1,-1)}) \\ q^{2J_3} &\equiv T_{(-2,0)} \quad q^{-2J_3} \equiv T_{(2,0)}. \end{aligned} \quad (14)$$

Using (12), one can show that

$$\begin{aligned} q^{J_3} J_\pm q^{-J_3} &= q^{\pm 1} J_\pm \\ [J_+, J_-] &= [2J_3]_q \end{aligned} \quad (15)$$

where $[x] \equiv (q^x - q^{-x}) / (q - q^{-1})$. These are the defining relations of the quantum algebra $sl_q(2)$, and the J 's constructed according to (14) are just the generators of this algebra.

We shall now show that the vacua form the basis of the representation of the $sl_q(2)$ algebra. To do this, we need to know the actions of T_n on the state $|\mathbf{h}_a\rangle$. From the definition of T_n (11) and the WH algebra (5), we find

$$T_n |\mathbf{h}_a\rangle = q^{-n_1 n_2 / 2} e^{i(n_2 l_J \cdot \gamma_1 - n_1 l_I \cdot \gamma_2 + n_1 h_a^I)} |\mathbf{h}_a - 2\pi n_2 l_J\rangle. \quad (16)$$

In obtaining (16), use has been made of the fact that the I th component of l_J is K_{IJ}^{-1} .

With the help of (16), it is easy to obtain the actions of the J 's on the state vectors $|\mathbf{h}_a\rangle$:

$$\begin{aligned} J_+|\mathbf{h}_a\rangle &= e^{i\mathbf{l}_J\cdot\boldsymbol{\gamma}_1}[(2\pi K_{IJ}^{-1})^{-1}(\mathbf{h}_a^I - \mathbf{l}_I \cdot \boldsymbol{\gamma}_2) - \frac{1}{2}]_q |\mathbf{h}_a - 2\pi\mathbf{l}_J\rangle \\ J_-|\mathbf{h}_a\rangle &= -e^{-i\mathbf{l}_J\cdot\boldsymbol{\gamma}_1}[(2\pi K_{IJ}^{-1})^{-1}(\mathbf{h}_a^I - \mathbf{l}_I \cdot \boldsymbol{\gamma}_2) + \frac{1}{2}]_q |\mathbf{h}_a + 2\pi\mathbf{l}_J\rangle \\ q^{\pm 2J_3}|\mathbf{h}_a\rangle &= q^{\mp(\pi K_{IJ}^{-1})^{-1}(\mathbf{h}_a^I - \mathbf{l}_I \cdot \boldsymbol{\gamma}_2)} |\mathbf{h}_a\rangle. \end{aligned} \quad (17)$$

Since W_2^J induces a mapping among the vacua as mentioned before, the vacua \mathbf{h}_a will in general form a cyclic representation of $sl_q(2)$ [18]. Suppose K_{IJ}^{-1} is such that $q \equiv e^{2\pi i K_{IJ}^{-1}}$ is a r th root of unity, then the cyclic representation is irreducible, and is of dimension r . Highest weight representation, however, could sometimes be obtained with an appropriate choice of the vacuum angles, as discussed in [7].

We now apply these results to two cases relevant to the FQHE.

Case I. $K = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

A theory defined by this K matrix serves as an alternative way of describing the first daughter state in the FQHE. This K gives

$$K^{-1} = \frac{1}{5} \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \quad \det K = 5. \quad (18)$$

The \mathbf{h}_a for the vacua in (7) is chosen to be

$$\mathbf{h}_a = \frac{2\pi a}{5} \mathbf{1} \quad (a = 0, \dots, 4). \quad (19)$$

For simplicity, we label the vacua $|\mathbf{h}_a\rangle$ by $|a\rangle$. They satisfy $|a+5\rangle = |a\rangle$. Vectors \mathbf{l}_I 's are given by

$$\mathbf{l}_1 = \frac{1}{5} \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad \mathbf{l}_2 = \frac{1}{5} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (20)$$

and

$$\mathbf{h}_a - 2\pi\mathbf{l}_I \sim \mathbf{h}_{a+2}. \quad (21)$$

From (10), we have,

$$\begin{aligned} W_1^I|a\rangle &= e^{-i\mathbf{l}_I\cdot\boldsymbol{\gamma}_2 + 2\pi i a/5} |a\rangle \\ W_2^I|a\rangle &= e^{+i\mathbf{l}_I\cdot\boldsymbol{\gamma}_1} |a+2\rangle. \end{aligned} \quad (22)$$

As $M = 2$ in this case, there are altogether four possible ways of forming the T operators according to (11), and therefore there exist four possible sets each of the quantum w_∞ and $sl_q(2)$ algebras. Note here that all the q 's formed from the four K_{IJ}^{-1} are primitive r th root of unity. Suppose we form the operators T_n with $I = 1$ and $J = 2$. Then $K_{12}^{-1} = -\frac{2}{5}$ and $q = e^{-4\pi i/5}$. The actions of $J_{\pm,3}$ as defined in (14) on the vacuum states are evaluated to be

$$\begin{aligned} J_+|a\rangle &= e^{i\beta_1/5} \left[-\frac{a}{2} + \frac{\alpha_2}{4\pi} - \frac{1}{2} \right]_q |a+2\rangle \\ J_-|a\rangle &= -e^{-i\beta_1/5} \left[-\frac{a}{2} + \frac{\alpha_2}{4\pi} + \frac{1}{2} \right]_q |a-2\rangle \\ q^{\pm 2J_3}|a\rangle &= q^{\pm 2((a/2) - (\alpha_2/4\pi))} |a\rangle \end{aligned} \quad (23)$$

where $\alpha_i \equiv 3\gamma_i^{(1)} - 2\gamma_i^{(2)}$ and $\beta_i \equiv -2\gamma_i^{(1)} + 3\gamma_i^{(2)}$.

It turns out that in this case there is, in addition to the four possible sets of J 's mentioned before, another set of generators of $sl_q(2)$ algebra. This is seen as follows. In [16] it is shown that, under appropriate coupling with matter fields, the multiple CS theory is effectively equivalent to a single CS theory with an effective Chern–Simons coefficient given by $\kappa_{\text{eff}}^{-1} = \sum_{I,J} K_{IJ}^{-1}$. κ_{eff} is in general a rational number, $\kappa_{\text{eff}} = p/q$ (p and q are two mutually prime integers), even if all K_{IJ} 's are integers. The effective Wilson line operators \bar{W}_i are related to the W_i^I in the multiple CS theory by $\bar{W}_i = W_i^{(1)} W_i^{(2)}$, and satisfy the WH algebra:

$$\bar{W}_1 \bar{W}_2 = \bar{q}^{-1} \bar{W}_2 \bar{W}_1 \quad \bar{q} \equiv e^{2\pi i/\kappa_{\text{eff}}}. \quad (24)$$

In the present case, one has $\kappa_{\text{eff}} = \frac{5}{2}$. The actions of \bar{W} 's on $|a\rangle$ are

$$\begin{aligned} \bar{W}_1 |a\rangle &= e^{-i(\lambda_2 - 4\pi a)/5} |a\rangle \\ \bar{W}_2 |a\rangle &= e^{i\lambda_1/5} |a-1\rangle \end{aligned} \quad (25)$$

where $\lambda_j \equiv \gamma_j^{(1)} + \gamma_j^{(2)}$. One can now define a new set of $sl_{\bar{q}}(2)$ generators \bar{J} 's according to (14), with the W 's replaced by the \bar{W} 's. The actions of \bar{J} are found to be

$$\begin{aligned} \bar{J}_+ |a\rangle &= e^{i\lambda_1/5} \left[a - \frac{\lambda_2}{4\pi} - \frac{1}{2} \right]_{\bar{q}} |a-1\rangle \\ \bar{J}_- |a\rangle &= -e^{-i\lambda_1/5} \left[a - \frac{\lambda_2}{4\pi} + \frac{1}{2} \right]_{\bar{q}} |a+1\rangle \\ \bar{q}^{\pm 2J_3} |a\rangle &= \bar{q}^{\mp 2(a - \lambda_2/4\pi)} |a\rangle. \end{aligned} \quad (26)$$

These expressions are precisely those obtained in [5, 7] for the case of single CS theory.

Case II. $K = \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}$.

This case corresponds to a filling factor $\frac{1}{3}$ in the FQHE. We have in this case

$$K^{-1} = \frac{1}{6} \begin{pmatrix} 3 & -3 \\ -3 & 5 \end{pmatrix} \quad \det K = 6 \quad \kappa_{\text{eff}} = 3. \quad (27)$$

A choice $\{\mathbf{h}_a\}$ for the vacua can be chosen to be

$$\mathbf{h}_a = \frac{\pi a}{3} \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad (a = 0 \sim 5). \quad (28)$$

We note that $|a+6\rangle = |a\rangle$. This time we have

$$\mathbf{l}_1 = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \mathbf{l}_2 = \frac{1}{6} \begin{pmatrix} -3 \\ 5 \end{pmatrix} \quad (29)$$

so that

$$\begin{aligned} \mathbf{h}_a - 2\pi \mathbf{l}_1 &\sim \mathbf{h}_{a-3} \\ \mathbf{h}_a - 2\pi \mathbf{l}_2 &\sim \mathbf{h}_{a+1}. \end{aligned} \quad (30)$$

Hence the actions of Wilson line operators are given by (we set the vacuum angles $\gamma_i = 0$ for simplicity)

$$\begin{aligned} W_1^{(1)} |a\rangle &= e^{i\pi a} |a\rangle \\ W_1^{(2)} |a\rangle &= e^{-5i\pi a/3} |a\rangle \\ W_2^{(1)} |a\rangle &= |a-3\rangle \\ W_2^{(2)} |a\rangle &= |a+1\rangle. \end{aligned} \quad (31)$$

As discussed previously, since only $q \equiv e^{2\pi i K_{IJ}^{-1}}$, with $K_{IJ} = K_{22}$, is a primitive r th root of unity, the six states only form the basis of an irreducible cyclic representation of a $sl_q(2)$ algebra for the J 's constructed by $W_1^{(2)}W_2^{(2)}$. The actions of $J_{\pm,3}$ so constructed on the vacuum states are evaluated to be

$$\begin{aligned} J_+|a\rangle &= -[a + \frac{1}{2}]_q|a+1\rangle \\ J_-|a\rangle &= [a - \frac{1}{2}]_q|a-1\rangle \\ q^{\pm 2J_3}|a\rangle &= q^{\pm 2a}|a\rangle. \end{aligned} \quad (32)$$

We note here that, unlike case I, the vacua $|a\rangle$ in this case do not form the basis of irreducible cyclic representation for operators \bar{J} obtained from the $\bar{W}_i = W_i^{(1)}W_i^{(2)}$. It is easily checked that $\bar{q} = e^{2\pi i/3}$ ($\bar{q}^3 = 1$). Thus there are only three inequivalent states in the irreducible cyclic representation of the $sl_{\bar{q}}$ algebra generated by the \bar{J} 's. This is related to the fact, as discussed in [16], that there exist only three inequivalent states in the effective single CS theory with $\kappa_{\text{eff}} = 3$. The actions of \bar{W}_2 on $|a\rangle$ are $\bar{W}_2|a\rangle = |a-2\rangle$, which separate the six states into two groups: $\{|a\rangle; a = 0, 2, 4\}$ and $\{|a\rangle; a = 1, 3, 5\}$. On the other hand, by a similar argument given in [16], one can check that the states $|a\rangle$ and $|a+3\rangle$ are equivalent in the effective theory. That means the states $\{|0\rangle, |3\rangle\}$, $\{|1\rangle, |4\rangle\}$ and $\{|2\rangle, |5\rangle\}$ correspond to the three distinct states in the effective theory. The Wilson line operator \bar{W}_2 , and hence the \bar{J}_{\pm} 's, map states among these three groups.

Finally, we remark that the above discussions and results can be directly carried over to the Maxwell–Chern–Simons theory with multiple kinds of CS fields and the related fractional quantum Hall theory on a torus [17]. When the Maxwell terms are included, the relevant operators are the so-called magnetic translation operators. Ground states form the basis of an algebra satisfied by these translation operators, which are precisely the WH group (5) obeyed by the W_j^I 's. Thus, while the actual forms of the ground states differ in the two theories, the algebraic structures are exactly the same. So they also share the same quantum group structures discussed in this letter.

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